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On The Stability of Dufour- Driven Generalized Hydromagnetic Double-Diffusive Shear Flows

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ABSTRACT: The present paper investigates the stability of Dufour–driven generalized hydromagnetic double-diffusive shear flows. The physical configuration is that of a horizontal layer of an incompressible inviscid heat conducting fluid of zero electrical resistivity in which there is a differential streaming U(z) in the horizontal direction and density variation $\rho_0 f(z)$ in the vertical direction while the entire system is confined between two horizontal boundaries of different but uniform temperature and concentration with the temperature and the concentration of the lower boundary greater than that of the upper one or vice-versa in the presence of a uniform horizontal magnetic field, ρ_0 being a positive constant having the dimension of density and U(z) and $f(z)$ being continuous functions of the vertical coordinate z with $\frac{df}{dz} < 0$ everywhere in the flow domain. Sufficient conditions are derived for overstability to be valid and bounds are presented for an arbitrary unstable mode of the system for the cases when the temperature and the concentration make opposing contributions to the vertical density gradient.
KEYWORDS: Chandersekher number; Double-diffusive convection; Dufour–effect; Non-homogeneous shear flows

I. INTRODUCTION

The stability of parallel shear flow of an inviscid non-homogeneous fluid with stable density stratification to infinitesimal non-divergent disturbances has pervaded the scientific literature in the recent past on account of its importance in the fields of meteorology and oceanography etc. For a broad view of the subject one may refer to the fundamental works of Taylor [1], Goldstein [2], Drazin [3], Miles [4], Howard [5] and others on the stability of non-homogeneous shear flows. In the mathematical model of the problem considered by these authors, the fluid is taken to be initially non-homogeneous without assigning any reason for the cause of this initial non-homogeneity. However, the initial non-homogeneity may be due to variable temperature or concentration or some other cause. Diffusion effects which tend to produce these changes in the density of an individual fluid particle in the course of motion are ignored in these investigations. Therefore, it became important to investigate the problem by retaining the initial non-homogeneity and also taking into account the diffusion effects. Gupta et.al. [6] investigated the problem by taking into account the changes in density due to thermal effects and referred to the problem as the problem of generalized thermal shear flows. In order to make the model of Gupta et.al. [6] more realistic as regards its applications in the fields of oceanography etc. Gupta et.al. [7] further investigated the problem by taking into account the changes in density due to thermal and concentration effects and referred to the problem as the problem of generalized thermohaline (Double-diffusive) shear flows.

The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour effects [8, 9]. An externally imposed temperature gradient produces a chemical potential gradient and the phenomenon known as the Soret effect, arises when the mass flux contains a term that depends upon the temperature gradient. The analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. Although it is clear that the thermosolutal and Soret-Dufour problems are quite closely related, their relationship has never been carefully elucidated. They are in fact, formally identical and this is done by means of a linear transformation that takes the equations and boundary conditions for the latter problem into those for the former. Mohan [10] mollified the nastily behaving governing equations of Dufour-driven thermosolutal convection of the Veronis

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[11] type by the construction of an appropriate linear transformation and extended the result of Banerjee et.al. [12] concerning the linear growth rate and the behavior of oscillatory motion. Mohan et al. [13] investigated the problem of generalized hydromagnetic thermohaline shear flows and derived sufficient conditions for overstability to be valid.

In the present paper we investigate the problem of Dufour-driven generalized hydromgnetic double-diffusive shear flows. Sufficient conditions are derived for overstability to be valid and bounds are presented for an arbitrary unstable mode of the system for the cases when the temperature and the concentration make opposing contributions to the vertical density gradient. The problem is completely solved at the marginal state when the basic velocity profile is linear and the diffusion of the dissolved solute is almost comparable than the diffusion of heat. A first approximation to the solution shows that as the initial density distribution increases, the Rayleigh number must also increase, a result which one would expect on physical grounds also.

II. MATHEMATICAL FORMULATION AND ANALYSIS

The relevant governing equations and boundary conditions of Dufour-driven double-diffusive shear flows wherein a uniform horizontal magnetic field is superimposed are given by [6]

$$(U-C)^2(D^2-a^2)w-(U-C)\,(D^2U)w-Q(D^2-a^2)w+R_3w=iR_1a(U-C)\theta-iR_Sa(U-C)\phi \ , \tag{1}$$

$$\{D^2-a^2-ia(U-C)\}\theta+\gamma\, R_0(D^2-a^2)\phi=-w \tag{2}$$

$$\left\{D^2-a^2-\frac{ia}{\tau}(U-C)\right\}\phi=-\frac{w}{\tau}, \tag{3}$$

where

$$C=\frac{i\sigma}{a}, R_1=\frac{g\alpha\beta d^4}{K_T^2}, R_S=\frac{g\alpha\beta d^4}{K_T^2}, R_3=\hat{R}_2 N^2, \hat{R}_2=\frac{\rho_0 d^4}{K_T{}^2}, R_0=\frac{\beta'}{\beta},$$

$$\gamma=\frac{\gamma_1}{K_T}\ \textit{is the Dufour number}, R_0=\frac{\beta'}{\beta},$$

and $N^2=-\dfrac{g\dfrac{df}{dz}}{\rho_0}$ is the Brunt-Vaisala frequency. In the above equations R_1 is thermal Rayleigh number, R_s is concentration Rayleigh number, K_T is the thermal diffusivity, K_S is the mass diffusivity, $\gamma_1=\dfrac{D_{01}}{c_v}$ is the Dufour coefficient, and the various other symbols have their usual meanings.

The solution of the Eqs. (1) - (3) must be sought subject to the following boundary conditions:

$$W=0=\theta=\phi \quad at \quad z=0 \quad and \quad z=1 \tag{4}$$

Equations (1) – (3) together with the boundary conditions (4) present an eigenvalue problem for $C(=C_r+iC_i)$, for given values of the other parameters and a given state of the system is stable, neutral or unstable provided C_i is negative, zero or positive respectively. Further, if $C_i=0$ implies that $C_r=0$ for every wave number a, then the principle of exchange of stabilities (PES) is valid, otherwise we have overstability at least when instability sets in a certain modes. It is to be noted that the inclusion of the convective effects of heat and mass transfer make the definitions of stable, neutral and unstable modes distinctly clear in the sense that the existence of a stable mode no longer implies the existence of an unstable mode etc., as is there in the classical instability problem of heterogeneous shear flows

We now prove the following theorems:

Theorem 1: If (C, w, θ, ϕ), C = C_r + i C_i is a solution of Eqs. (1) – (4) with $R_1>0, R_S>0, \gamma>0, 0<\tau<1$, Q > 0 and $Q<U^2_{min}$ and

i) $UD^2U>0,\ \forall\ z\in[0,1]$,

ii)

$$R_S\le\left\{\frac{|UD^2U|}{2}-\frac{Q}{U}\frac{(D^2U)}{2}-R_3\right\}_{min}-\frac{R_1R_0\gamma}{1-\tau},$$

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

Proof. Using the transformations

$$\left.\begin{aligned}\tilde{\theta} &= \frac{1-\tau}{R_0\gamma}\theta + \phi \\ \tilde{\phi} &= \phi \\ \tilde{w} &= w\end{aligned}\right\},$$

(5) the system of Eqs. (1) - (3) together with the boundary condition (4) assume the following forms:

$$(U-C)^2\left(D^2-a^2\right)w-(U-C)\left(D^2U\right)w- Q\left(D^2-a^2\right)w+R_3w = iR_1'a(U-C)\theta - iR_S'a(U-C)\phi,$$
(6)

$$\left\{D^2-a^2-ia(U-C)\right\}\theta = -Mw,$$
(7)

$$\left\{D^2-a^2-\frac{ia}{\tau}(U-C)\right\}\phi = -\frac{w}{\tau},$$
(8)

with

$$W=0=\theta=\phi \quad at \quad z=0 \quad and \quad z=1 \qquad (9)$$

where

$$R' = \frac{R_1R_0\gamma}{1-\tau} \ (Modified\ thermal\ Rayliegh\ number),$$

$$R_S' = R_S + R'(Modified\ concentration\ Rayliegh\ number),$$

$$M = 1+\frac{1-\tau}{R_0\gamma},$$

The sign tilde (~) has been omitted for simplicity.

If possible, let $C_i = 0 \Rightarrow C_r = 0, \ \forall \ a$ so that $C = 0$ is allowed by the governing equations and boundary conditions.

Then from equations (6) – (8), we have

$$U^2\left(D^2-a^2\right)w-U\left(D^2U\right)w-Q\left(D^2-a^2\right)w+R_3w = iR_1'a\theta U - iR_S'aU\phi, \qquad (10)$$

$$\left\{D^2-a^2-iaU\right\}\theta = -Mw,$$
(11)

$$\left\{D^2-a^2-\frac{iaU}{\tau}\right\}\phi = -\frac{w}{\tau}.$$
(12)

In view of condition (i) of the theorem

$U \neq 0, \ \forall \ z \in [0,1]$,

So that Eq. (10) can also be written as

$$U\left(D^2-a^2\right)w-\left(D^2U\right)w-Q\left(D^2-a^2\right)\frac{w}{U} + R_3\frac{w}{U} = iR_1'a\theta - iR_S'a\phi$$
(13)

Multiplying Eqs. (13), (11) and (12) by $W^*, \frac{-iR_1'a\theta^*}{M}$ and $i\tau R_S'a\phi^*$ (*indicates complex conjugation), respectively, integrating over the vertical range of z by parts appropriately, using the boundary conditions (9), we get

$$\int_0^1 U\left(|Dw|^2+a^2|w|^2\right)dz + \int_0^1 w^*DU\ Dw\ dz + \int_0^1\left(D^2U\right)|w|^2dz + \frac{R_1'}{M}a^2\int_0^1 U|\theta|^2dz + iR_S'a\tau\int_0^1\left(|D\phi|^2+a^2|\phi|^2\right)dz + -i\frac{R_1'}{M}a\int_0^1\left(|D\theta|^2+a^2|\theta|^2\right)dz - Q\int_0^1\frac{1}{U}\left(|Dw|^2+a^2|w|^2\right)dz + Q\int_0^1\frac{w\,Dw^*DU}{U^2}dz = \int_0^1\frac{R_3}{U}|w|^2dz + 2iR_S'\ a\operatorname{Re}\int_0^1\phi w^*\,dz - 2iR_1'\,a\operatorname{Re}\int_0^1\theta w^*dz \qquad (14)$$

where Re stands for the real part.

Integrating the second term on the left hand side of Eq. (14) by parts once and using boundary conditions (9), we get

$$\operatorname{Re}\left(\int_0^1 W^*DUDWdz\right) = \frac{-1}{2}\int_0^1 D^2U\,|W|^2\,dz.,$$
(15)

Equating the real parts of Eq. (14) and making use of Eq. (15), we have

$$\int_0^1 U\left(|Dw|^2+a^2|w|^2\right)dz + \frac{1}{2}\int_0^1\left(D^2U\right)\left(|w|^2\right)dz + \frac{R_1'}{M}a^2\int_0^1 U|\theta|^2dz - \frac{Q}{2}\int_0^1\frac{|w|^2D^2U}{U^2}dz +$$

$$+Q\int_0^1 \frac{|w|^2 (DU)^2}{U^3} dz - Q\int_0^1 \frac{1}{U}\left(|Dw|^2 + a^2|w|^2\right)dz$$

$$= \int_0^1 \frac{R_3}{U}|w|^2 dz + R'_S a^2 \int_0^1 U|\phi|^2 dz \tag{16}$$

Multiplying Eq. (12) by its complex conjugate and integrating over the vertical range of z, we get

$$\int_0^1 \frac{1}{U}\left|\left(D^2 - a^2\right)\phi\right|^2 dz + \frac{a^2}{\tau^2}\int_0^1 U|\phi|^2 dz = \frac{1}{\tau^2}\int_0^1 \frac{1}{U}|w|^2 dz. \tag{17}$$

Condition (i) of the theorem implies that either

(a) $U > 0$, $D^2U > 0$ or

(b) $U < 0$, $D^2U < 0$, $\forall$ $z \in [0, 1]$.

If (a) holds, then equation (17) gives

$$\int_0^1 U|\phi|^2 dz < \frac{1}{a^2}\int_0^1 \frac{1}{U}|w|^2 dz \tag{18}$$

Using inequality (18) in Eq. (16), we get

$$\int_0^1 U\left(|Dw|^2 + a^2|w|^2\right)dz + \frac{1}{2}\int_0^1 \left(D^2U\right)\left(|w|^2\right)dz +$$

$$\frac{R'_1}{M} a^2 \int_0^1 U|\theta|^2 dz - \frac{Q}{2}\int_0^1 \frac{|w|^2 D^2U}{U^2} dz +$$

$$+Q\int_0^1 \frac{|w|^2 (DU)^2}{U^3} dz - Q\int_0^1 \frac{1}{U}\left(|Dw|^2 + a^2|w|^2\right)dz$$

$$-\left\{\int_0^1 \frac{R_3}{U}|w|^2 dz + R'_S \int_0^1 \frac{1}{U}|w|^2 dz\right\} < 0$$

or

$$\int_0^1 \left(U - \frac{Q}{U}\right)\left\{|Dw|^2 + a^2|w|^2\right\}dz + \frac{R'_1}{M} a^2 \int_0^1 |\theta|^2 U dz$$

$$+Q\int_0^1 \frac{|w|^2 (DU)^2}{U^3} dz$$

$$+\int_0^1 \frac{1}{U}\left(\frac{UD^2U}{2} - \frac{\left(D^2U\right)Q}{2U} - R_3 - R'_S\right)|w|^2 dz < 0 \tag{19}$$

If (b) holds, it is easily seen that inequality (19) assumes the form

$$\int_0^1 \left(|U| - \frac{Q}{|U|}\right)\left\{|Dw|^2 + a^2|w|^2\right\}dz$$

$$+\frac{R'_1}{M} a^2 \int_0^1 |\theta|^2 |U| dz + Q\int_0^1 \frac{|w|^2 (DU)^2}{|U|^3}$$

$$+\int_0^1 \frac{1}{|U|}\left(\frac{|U||D^2U|}{2} - \frac{\left(|D^2U|\right)Q}{2|U|} - R_3 - R'_S\right)|w|^2 dz < 0 \tag{20}$$

Inequalities (19) and (20) obviously cannot hold under conditions (i) and (ii) of the theorem.

Hence, in the condition of the theorem $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

This completes the proof of the theorem.

The essential contents of Theorem 1, from the point of view of hydrodynamic instability, is that an arbitrary neutral mode in the problem of generalized hydromagnetic Dufour-driven double-diffusive shear flows of Veronis [11] ($R_1 > 0$, $R_S > 0$) type is definitely not non-oscillatory ($C_r = 0$) in character, i.e. PES is not valid if $UD^2U > 0$, everywhere in [0,1] and

(i) $Q < U^2_{\min}$,

(ii)

$$R_S \le \left\{\frac{|UD^2U|}{2} - \frac{Q}{U}\frac{\left(D^2U\right)}{2} - R_3\right\}_{\min} - \frac{R_1 R_0 \gamma}{1-\tau},$$

SPECIAL CASES: It follows from Theorem 1 that PES is not valid for

(i) Dufour-driven generalized double-diffusive shear flows (Q = 0) if

$$U\,D^2U > 0,\ \forall\ z \in [0,1]$$

and

$$R_S \le \left\{\frac{|UD^2U|}{2} - R_3\right\}_{\min} - \frac{R_1 R_0 \gamma}{1-\tau},$$

(ii) Dufour-driven double-diffusive shear flows (R_3 = Q= 0) if

$$U\,D^2U > 0,\ \forall\ z \in [0,1]$$

and $R_S \le \left\{\frac{|UD^2U|}{2}\right\}_{\min} - \frac{R_1 R_0 \gamma}{1-\tau},$

Further, taking γ =0 in Eq. (2) and proceeding exactly as in Theorem1, the following results can easily be derived.

(iii) Generalized hydromagnetic double-diffusive shear flows if

$U\ D^2U > 0,\ \forall\ z \in [0,1]$

and $R_S \le \left\{ \frac{|UD^2U|}{2} - \frac{Q}{U}\frac{(D^2U)}{2} - R_3 \right\}_{\min}$,

(iv) Hydromagnetic double-diffusive shear flows of Veronis type ($R_3 = 0$, $R_1 > 0$, $R_S > 0$) if

$U\ D^2U > 0,\ \forall\ z \in [0,1]$

and $R_S \le \left\{ \frac{UD^2U}{2} - \frac{QD^2U}{2U} \right\}_{\min}$.

(v) Double-diffusive shear flows of Veronis type ($R_1 >$ 0, $R_S > 0$, $R_3 = 0 = Q$) if

$U\ D^2U > 0,\ \forall\ z \in [0,1]$ and

$R_S \le \left\{ \frac{UD^2U}{2} \right\}_{\min}$.

Theorem 2: If (C, w, θ, ϕ), $C = C_r + iC_i$ is a solution of Eqs. (6) – (9) with $R_1 > 0$, $R_S > 0$, $Q > 0$ and $Q < U^2_{\min}$ and

(i) (a) $U > 0$ and $D^2U \le 0,\ \forall\ z \in [0,1]$, or

(b) $U < 0$ and $D^2U \ge 0,\ \forall\ z \in [0,1]$,

(ii)

$$R'_S \le \left\{ \left\langle \pi^2 U^2 - \frac{|U\ D^2U|}{2} \right\rangle \left(1 - \frac{Q}{U^2}\right) - R_3 \right\}_{\min} ,$$

then $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

Proof: If possible, let $C_i = 0 \Rightarrow Cr = 0\ \forall$ a, so that C = 0 is allowed by the governing equations and boundary conditions.

Proceeding exactly as in Theorem1 upon considering the case (i) (a), we have

$$\int_0^1 U\left(|Dw|^2 + a^2|w|^2\right)dz + \frac{R'_1}{M}a^2\int_0^1 U|\theta|^2 dz +$$
$$\frac{Q}{2}\int_0^1 \frac{|w|^2|\ D^2U\ |}{U^2}dz + Q\int_0^1 \frac{|w|^2(DU)^2}{U^3}dz$$
$$= \int_0^1 \frac{R_3}{U}|w|^2 dz + R'_S a^2 \int_0^1 U|\phi|^2 dz +$$
$$Q\int_0^1 \frac{1}{U}\left(|Dw|^2 + a^2|w|^2\right)dz + \frac{1}{2}\int_0^1 \left(|\ D^2U\ |\right)\left(|w|^2\right)dz$$

(21)

Equation (21) together with inequality (18), yields

$$\int_0^1 U\left(|Dw|^2 + a^2|w|^2\right)dz + \frac{R'_1}{M}a^2\int_0^1 U|\theta|^2 dz +$$
$$\frac{Q}{2}\int_0^1 \frac{|w|^2|D^2U|}{U^2}dz + Q\int_0^1 \frac{|w|^2(DU)^2}{U^3}dz$$
$$< \int_0^1 \frac{R_3}{U}|w|^2 dz + R'_S\int_0^1 \frac{1}{U}|w|^2 dz$$
$$+ Q\int_0^1 \frac{1}{U}\left(|Dw|^2 + a^2|w|^2\right)dz + \frac{1}{2}\int_0^1 \left(|D^2U|\right)\left(|w|^2\right)dz$$

(22)

Now since $U > 0$, $U^2 - Q > 0$, $\forall\ z \in [0,1]$, we have

$$\int_0^1 \left(\frac{U^2 - Q}{U}\right)|Dw|^2 dz \ge \left(\frac{U^2 - Q}{U}\right)_{\min}\int_0^1 |Dw|^2 dz$$

,

which upon using the Rayleigh-Ritz inequality [Schultz [14]], namely,

$$\int_0^1 |Df_1|^2 dz \ge \pi^2 \int_0^1 |f_1|^2 dz ,$$

(23)

where $f_1(0) = 0 = f_1(1)$ with $f_1 = w$ gives

$$\int_0^1 U|Dw|^2 dz \ge \pi^2 U_{\min}\int_0^1 |w|^2 dz.$$

(24)

Using inequality (24) in inequality (22), we get

$$\int_0^1 \left\{ \begin{array}{l} \pi^2 \left\langle \frac{U^2\left(1 - \frac{Q}{U^2}\right)}{U} \right\rangle_{\min} - \\ \frac{1}{2}\left(1 - \frac{Q}{U^2}\right)|D^2U| - \frac{R_3}{U} - \frac{R'_S}{U} \end{array} \right\}|w|^2 dz +$$
$$+ \frac{R'_1}{M}a^2\int_0^1 U|\theta|^2 dz + Q\int_0^1 |w|^2\frac{(DU)^2}{U^3}dz < 0$$

or

$$\int_0^1 \frac{1}{U}\left\{\begin{array}{l} U\pi^2\left\langle \dfrac{U^2\left(1-\dfrac{Q}{U^2}\right)}{U}\right\rangle_{min} - \\ \dfrac{1}{2}\left(1-\dfrac{Q}{U^2}\right)\left|UD^2U\right| - R_3 - R_S' \end{array}\right\}|w|^2 dz + $$

$$+\frac{R_1'}{M}a^2\int_0^1 U|\theta|^2 dz + Q\int_0^1 |w|^2 \frac{(DU)^2}{U^3} dz < 0 .$$

(25)

Similarly if i(b) i.e. $U < 0$ and $D^2U \geq 0$, $\forall$ $z \in [0,1]$, then it is easily seen that inequality (25) assumes the form

$$\int_0^1 \frac{1}{|U|}\left\{\begin{array}{l} |U|\pi^2\left\langle \dfrac{U^2\left(1-\dfrac{Q}{U^2}\right)}{|U|}\right\rangle_{min} - \\ \dfrac{1}{2}\left(1-\dfrac{Q}{U^2}\right)\left|UD^2U\right| - R_3 - R_S' \end{array}\right\}|w|^2 dz + $$

$$+\frac{R_1'}{M}a^2\int_0^1 |U||\theta|^2 dz + Q\int_0^1 |w|^2 \frac{(DU)^2}{|U|^3} dz < 0 .$$

(26)

Inequalities (25) – (26) obviously cannot hold under the conditions of the theorem.

Hence, under the conditions of the theorem $C_i = 0 \Rightarrow C_r \neq 0$ for some wave number a.

This completes the proof of the theorem.

The essential content of Theorem 2, from the point of view of hydrodynamic instability, is that an arbitrary neutral mode in the problem of Dufour-driven generalized hydromagnetic double-diffusive shear flows of Veronis' [11] ($R_1 > 0$, $R_S > 0$) is definitely not non-oscillatory ($C_r = 0$) in character, i.e. PES is not valid if $U > 0$ and $D^2U \leq 0$, everywhere in [0, 1] and $U < 0$ and $D^2U \geq 0$, everywhere in [0, 1] and

(i) $Q < U^2_{min}$,

(ii)

$$R_S' \leq \left\{\left\langle \pi^2U^2 - \frac{|U\ D^2U|}{2}\right\rangle\left(1-\frac{Q}{U^2}\right) - R_3\right\}_{min}$$

or

$$R_S \leq \left\langle \left\{\Pi^2 U^2 - \frac{1}{2}|UD^2U|\right\}\left(1-\frac{Q}{U^2}\right) - R_3 \right\rangle_{min} - \frac{R_1R_0\gamma}{1-\tau}$$

SPECIAL CASES: It follows from Theorem 2 that the PES is not valid for

i) Dufour-driven generalized double-diffusive shear flows (Q=0) if $U > 0$, $D^2U \leq 0$, or $U < O$, $D^2U \geq 0$, $\forall$ $z \in [0,1]$ and

$$R_S \leq \left\{\left\langle \pi^2U^2 - \frac{|U\ D^2U|}{2}\right\rangle - R_3\right\}_{min} - \frac{R_1R_0\gamma}{1-\tau}$$

(ii) Dufour-driven double-diffusive shear flows (R_3 = Q= 0) if

$D^2U \leq 0$, or $U < O$, $D^2U \geq 0$, $\forall$ $z \in [0,1]$

$$R_S \leq \left\{\left\langle \pi^2U^2 - \frac{|U\ D^2U|}{2}\right\rangle\right\}_{min} - \frac{R_1R_0\gamma}{1-\tau}$$

Further, taking $\gamma = 0$ in Eq. (2) and proceeding exactly as in Theorem 2, following results can easily be derived.

(iii) Generalized hydromagnetic double-diffusive shear flows if

$$U > 0, D^2U \leq 0 \;\; or\, U < 0, D^2U \geq 0 \;\forall\; z \in [0,1]$$

$$R_S \leq \left\{\left\langle \pi^2U^2 - \frac{|U\ D^2U|}{2}\right\rangle\left(1-\frac{Q}{U^2}\right) - R_3\right\}_{min}$$

(iv) Double-diffusive shear flows of Veronis type ($R_3 = 0 =$ Q, $R_1 > 0$, $R_S > 0$) if $U > 0$, $D^2U \leq 0$, or $U < 0$, $D^2U \geq 0$, $\forall$ $z \in [0,1]$

and $R_S \leq \left\{\pi^2U^2 - \frac{1}{2}\left|U\ D^2U\right|\right\}_{min}$.

(v) Hydromagnetic double-diffusive shear flows of Veronis type ($R_3 = 0$, $R_1 > 0$, $R_S > 0$) if

$U > 0$, $D^2U \leq 0$, or $U < 0$, $D^2U \geq 0$, $\forall$ $z \in [0,1]$

and

$$R_S \leq \left\{\left\langle \pi^2U^2 - \frac{1}{2}\left|U\ D^2U\right|\right\rangle\left(1-\frac{Q}{U^2}\right)\right\}_{min} .$$

Theorem 3: If (C, w, θ, ϕ), $C = C_r + iC_i$) is a solution of equations (6)-(9) with $R_1 < 0$, $R_S < 0$, $Q > 0$ and $Q < U^2_{min}$ and

(i) $U\ D^2U > 0$, $\forall$ $z \in [0,1]$,

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(ii)

$$|R_1| \le \frac{1-\tau}{M^2 R_0 \gamma}\left\{\frac{U\, D^2U}{2} - \frac{Q\, D^2U}{2U} - R_3\right\}_{\min},$$

then $C_i = 0 \Rightarrow C_r \ne 0$ for some wave number a.

Proof: Putting $R_1 = -|R_1|$ and $R_S = -|R_S|$, in Eq. (16), and using the inequality

$$\int_0^1 U|\theta|^2 dz < \frac{M^2}{a^2}\int_0^1 \frac{1}{U}|w|^2 dz.$$

(27)

which is derived from Eq. (7) in a manner similar to the derivation of inequality (18), and proceeding exactly as in Theorem 1, we get the required result.

This completes the proof of the Theorem.

Theorem 4: If (C, w, θ, ϕ), C = C_r + iC_i is a solution of equations (6)-(9) with $R_1 < 0$, $R_S < 0$, Q > 0 and Q < $U^2_{\min}$ and

(i) a) U > 0, $D^2U \le 0$, $\forall$ z $\in [0,1]$ or

b) U < 0, $D^2U \ge 0$, $\forall$ z $\in [0,1]$

(ii)

$$|R_1| \le \frac{1-\tau}{M^2 R_0\gamma}\left\{\left\langle \pi^2 U^2 - \frac{|U\, D^2U|}{2}\right\rangle\left(1-\frac{Q}{U^2}\right) - R_3\right\}_{\min},$$

then $C_i = 0 \Rightarrow C_r \ne 0$ for some wave number a.

Proof: Putting R_1 = -$|R_1|$ and R_S = - $|R_S|$, in Eq. (16), using inequalities (24) and (27) and proceeding exactly as in Theorem 2, we get the required result.

This compeltes the proof of the theorem.

The essential contents of Theorem 3 and Theorem 4, from the point of view of hydrodynamic instability are similar to that of Theorem1 and Theorem2 respectively. However, presently the problem is that of Dufour-driven generalized hydromagnetic double-diffusive shear flows of Stern's type [15] (R_1 < 0, R_S < 0).Further special cases of Theorem3 and 4 analogous to that of the earlier Theorems could be easily written down in the present case also.

Theorem 5: If (C, w, θ, ϕ), C = C_r + iC_i, C_i > 0 is a solution of equation (6) – (9) with $R_1 > 0$, R_S > 0 and Q > 0, then $C_i < \alpha$, where α is the positive root of the cubic

$$\pi^2 C_i^3 - \pi\, qC_i^2 - R_1' MC_i - \pi Qq = 0,$$

where q = ($|DU|$)$_{\max}$.

Proof: Since U – C $\ne$ 0, $\forall$ z $\in [0,1]$, therefore dividing Eq. (6) throughout by (U – C) and then proceeding as in Theorem 1, we get

$$\int_0^1 (U-C)\left(|Dw|^2 + a^2|w|^2\right)dz + \int_0^1 w^*(DU)(Dw)\,dz + \int_0^1 (D^2U)|w|^2\,dz + \frac{R_1'}{M}a^2\int_0^1 (U-C)|\theta|^2 dz + iR_S' a\tau\int_0^1\left(|D\phi|^2 + a^2|\phi|^2\right)dz - i\frac{R_1'}{M}a\int_0^1\left(|D\theta|^2 + a^2|\theta|^2\right)dz - Q\int_0^1 \frac{1}{(U-C)}\left(|Dw|^2 + a^2|w|^2\right)dz + Q\int_0^1 \frac{w\, Dw^* DU}{(U-C)^2}dz = \int_0^1 \frac{R_3}{(U-C)}|w|^2 dz + 2iR_S' a\,\mathrm{Re}\int_0^1 \phi w^*\,dz - 2iR_1' a\,\mathrm{Re}\int_0^1 \theta w^*\,dz, \quad (28)$$

where Re stands for real part.

Equating the imaginary parts of Eq. (28) and dividing the resulting equation throughout by C_i (> 0), we get

$$\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz + \frac{R_1'}{M}a^2\int_0^1|\theta|^2 dz + \frac{R_1' a}{C_i M}\int_0^1\left(|D\theta|^2 + a^2|\theta|^2\right)dz + \int_0^1\frac{R_3}{|U-C|^2}|w|^2 dz + \frac{2R_S' a}{C_i}\mathrm{Re}\int_0^1\left(\phi w^* dz\right) + Q\int_0^1\frac{1}{|U-C|^2}\left(|Dw|^2 + a^2|w|^2 dz\right)$$
$$= \mathrm{Im}\left(\frac{1}{C_i}\int_0^1 (DU)(w^*)(Dw)dz\right) + \mathrm{Im}\left(\frac{Q}{C_i}\int_0^1\frac{w\, Dw^*\, DU}{(U-C)^2}\right) + \frac{\tau R_S' a}{C_i}\int_0^1\left(|D\phi|^2 + a^2|\phi|^2\right)dz + R_S' a^2\int_0^1|\phi|^2 dz + \frac{2R_1' a}{C_i}\mathrm{Re}\left(\int_0^1 \theta w^* dz\right) \quad (29)$$

where Im stands for the imaginary part.

Using Eqs. (7) – (8), it follows that

$$\mathrm{Re}\left(\int_0^1 \theta w^* dz\right) = \frac{1}{M}\int_0^1\left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz \quad (30)$$

and

$$\mathrm{Re}\left(\int_0^1 \phi w^* dz\right) = \tau\int_0^1 \left(|D\phi|^2 + a^2|\phi|^2 + aC_i|\phi|^2\right)dz,$$ (31)

Using Eqs. (30)–(31) in Eq. (29) and simplifying the resulting equation, we get

$$\int_0^1 \left(|Dw|^2 + a^2|w|^2\right)dz + \frac{\tau R_S' a}{C_i}\int_0^1 \left(|D\phi|^2 + a^2|\phi|^2 + \frac{aC_i}{\tau}|\phi|^2\right)dz$$

$$+\int_0^1 \frac{R_3}{|U-C|^2}|w|^2 dz +$$

$$Q\int_0^1 \frac{1}{|U-C|^2}\left(|Dw|^2 + a^2|w|^2\right)dz = \frac{R_1' a}{MC_i}\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz$$

$$\mathrm{Im}\left(\frac{1}{C_i}\int_0^1 (DU)(w^*)(Dw)dz\right) + \mathrm{Im}\left(\frac{Q}{C_i}\int_0^1 \frac{(DU)(Dw^*)w}{(U-C)^2}dz\right)$$ (32)

Multiplying Eq. (7) by θ^*, integrating over the vertical range of z by parts once, using boundary conditions (9) and equating the real parts of the resulting equation, we get

$$\frac{1}{M}\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz = \mathrm{Re}\left(\int_0^1 w\theta^* dz\right)$$

$$\leq \int_0^1 |w||\theta| dz$$

$$\leq \left\{\int_0^1 |w|^2 dz\right\}^{1/2} \left\{\int_0^1 |\theta|^2 dz\right\}^{1/2}$$ (33)

It follows from inequality (33) that

$$\frac{1}{M}aC_i\int_0^1 |\theta|^2 dz < \left\{\int_0^1 |w|^2 dz\right\}^{1/2} \left\{\int_0^1 |\theta|^2 dz\right\}^{1/2}$$

i.e.

$$\left\{\int_0^1 |\theta|^2 dz\right\}^{1/2} < \frac{M}{aC_i}\left\{\int_0^1 |w|^2 dz\right\}$$

. (34)

Using inequality (34) in inequality (33), we get

$$\frac{1}{M}\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz < \frac{M}{aC_i}\int_0^1 |w|^2 dz,$$

which upon using inequality (23) with $f_1 = w$, gives

$$\int_0^1 \left(|D\theta|^2 + a^2|\theta|^2 + aC_i|\theta|^2\right)dz < \frac{M^2}{aC_i\pi^2}\int_0^1 |Dw|^2 dz$$ (35)

Further

$$\mathrm{Im}\left(\frac{1}{C_i}\int_0^1 (DU)(w^*)(Dw)dz\right) \leq \frac{1}{C_i}\int_0^1 |DU|\,|w|\,|Dw|\,dz$$

$$\leq \frac{q}{C_i}\int_0^1 |w|\,|Dw|\,dz$$

$$\leq \frac{q}{C_i}\left\{\int_0^1 |w|^2 dz\right\}^{1/2} \left\{\int_0^1 |Dw|^2 dz\right\}^{1/2},$$

which upon using inequality (23) with $f_1 = w$, gives

$$\mathrm{Im}\left(\frac{1}{C_i}\int_0^1 (DU)(w^*)(Dw)dz\right) \leq \frac{q}{C_i\pi}\int_0^1 |Dw|^2 dz,$$ (36)

where $q = (|DU|)_{max}$,

and

$$\mathrm{Im}\left(\frac{1}{C_i}\int_0^1 \frac{(Dw^*)(w)(DU)}{(U-C)^2}dz\right) \leq \frac{1}{C_i}\int_0^1 \frac{|Dw|\,|w|\,|DU|}{|U-C|^2}dz$$

$$\leq \frac{q}{C_i^2}\int_0^1 |w|\,|Dw|\,dz$$

$$\leq \frac{q}{C_i^2}\left\{\int_0^1 |w|^2 dz\right\}^{1/2} \left\{\int_0^1 |Dw|^2 dz\right\}^{1/2},$$

which upon using inequality (23) with $f_1 = w$, gives

$$\mathrm{Im}\left(\frac{Q}{C_i}\int_0^1 \frac{(Dw^*)(w)(DU)}{(U-C)^2}dz\right) \leq \frac{Qq}{C_i^2\pi}\int_0^1 |Dw|^2 dz$$ (37)

where $q = (|DU|)_{max}$.

Equation (32) upon using inequalities (35) – (37), gives

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$$\left\{1-\frac{R_1'M}{\pi^2C_i^2}-\frac{q}{\pi^2C_i}-\frac{Qq}{\pi C_i^3}\right\}\int_0^1|Dw|^2dz+a^2\int_0^1|w|^2dz+$$
$$\frac{\tau R_S'a}{C_i}\int_0^1\left[\left(|D\phi|^2+a^2|\phi|^2\right)+\frac{aC_i}{\tau}|\phi|^2\right]dz$$
$$+\int_0^1\frac{R_3}{|U-C|^2}|w|^2dz+Q\int_0^1\frac{1}{|U-C|^2}\left(|Dw|^2+a^2|w|^2\right)dz<0 \quad (38)$$

Since $a > 0$, $C_i > 0$, $\tau > 0$, $R_S > 0$ and $R_3 > 0$, therefore inequality (38) clearly implies that

$$\pi^2C_i^3-\pi\, qC_i^2-R_1'MC_i-\pi Qq<0 \ .$$

Hence, if α_1, α_2, α_3 are roots of this cubic, then $\alpha_1\alpha_2\alpha_3 = \pi$ Qq > 0 ⇒ cubic has one and only one positive root $\alpha_1 = \alpha$ (say). Thus above cubic yields $C_i < \alpha$.

This completes the proof of the theorem.

The essential content of Theorem 5 from the point of view of hydrodynamic instability is that the growth rate of an arbitrary unstable ($C_i > 0$) mode in the problem of Dufour-driven generalized hydromagnetic double-diffusive shear flows of Veronis' type ($R_1 > 0$, $R_S > 0$) is necessarily bounded with upper bound α.

Further, this result is uniformly valid for the problems of Dufour-driven hydromagnetic double-diffusive shear flows, generalized hydromagnetic double-diffusive shear flow, Dufour-driven generalized double-diffusive shear flows etc. of Veronis' type.

Theorem 6: If (C, w, θ, ϕ), $C = C_r + iC_i$, $C_i > 0$ is a solution of equations (6) – (9) with $R_1 < 0$, $R_S < 0$, $Q > 0$, then $C_i < \alpha$, α being positive root of cubic

$$\pi^2C_i^3-\pi\, qC_i^2-|R_S'|C_i-\pi Qq=0\ ,$$

where $q = (|DU|)_{max}$.

Proof: Putting $R_1 = |R_1|$ and $R_S = -|R_S|$ in Eq. (32), using the inequalities (35) and

$$\tau\int_0^1\left(|D\phi|^2+a^2|\phi|^2\right)dz+aC_i\int_0^1|\phi|^2<\frac{1}{aC_i\pi^2}\int_0^1|Dw|^2dz \quad (39)$$

which is derived from Eq. (8) in a manner similar to the derivation of inequality (34), we get the required result.

This completes the proof of the theorem.

The essential content of Theorem 6, from the point of view of hydrodynamic instability, is similar to that of Theorem5.However, the problem presently is that of Dufour –driven generalized hydromagnetic double-diffusive shear flows of Stern's type ($R_1 < 0$, $R_S < 0$).

REFERENCES

[1]. Taylor G.I., (1931), Effect of variation in density on the stability of superposed streams of fluid. Proc. Roy. Soc. (London), A 132, 499- 523.

[2]. Goldstein S., (1931), On the stability of superposed streams of fluid of different densities, Proc. Roy. Soc. (London) A 132, 524- 548.

[3]. Drazin P.G., (1958), The instability of a shear layer in an unbounded heterogeneous inviscid fluid, J. Fluid Mech., 4, 214-217.

[4]. Miles J.W., (1961), On the stability of heterogeneous shear flows, J. Fluid Mech., 10, 496-508.

[5]. Howard L.N.,. (1961), Note on a paper of John W. Miles', .J. Fluid Mech., 10, 509-512.

[6]. Gupta J.R., Banerjee Mihir, B., Pathania R.C. & Dube S.N., (1977), On the stability of generalized thermal shear flows, J. Math. Phys. Sci., 11(2), 165-172.

[7]. Gupta J.R., & Pathania R.C., (1988), On the stability of generalized thermohaline shear flows, Ganita, 39(2), 79-96.

[8]. Fitts D.D. (1962), NonequlibriumThermody namics: McGraw Hill,New York,

[9]. Groot S.R.de & Mazur. P. (1962), Nonequlibrium Thermodynamics, North Holland, Amsterdam,

[10]. Mohan H., (1998), Dufour-driven thermosolutal convection of the Veronis type, J.Math.Anal.Appl., 218, 569-580.

[11]. Veronis G., (1965), On finite amplitude instability in thermohaline convection, J. Mar. Res., 23, 1-17.

[12]. Banerjee M.B,.Katoch D.C,.Dube G.S. & .Banerjee K, (1981), Bounds for a growth rate of a perturbation in thermohaline convection, Proc.Roy.Soc.London, Ser. A 378, 301-304.

[13]. Mohan.H,Kumar.P.&Devi,Pushpa,(2004),On the stability of generalized hydromagnetic thermohaline shear flows,Studia Geotechnica et Mechanica,1-2,63-77.

[14]. Schultz. M.H., (1973), Sapline Analysis, Prantice Hall, Englewood Cliff, NJ.

[15]. Stern M.E., (1960), The salt fountain and thermohaline convection, Tellus.,12, 172-175.